THE GEODESIC LINES ON THE HELICOID

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Introduction. To find the geodesics on the helicoid we will first obtain them for a surface to which the helicoid is applicable and on which they are more easily obtained. Since lengths are preserved in the application of surfaces and since geodesics are shortest lines they must correspond on two applicable surfaces. Or, analytically, the geodesics will correspond on two applicable surfaces since only E, F, and G and their derivatives enter the differential equation of the geodesics.

It has been shown* that the helicoid whose equations are

$$x = u' \cos v', \quad y = u' \sin v', \quad z = av',$$

u' being the radius vector and v' the angle made by the x-axis and the projection of the radius vector on the xy-plane, is applicable to the catenoid whose equations are

$$x = u \cos v, \quad y = u \sin v, \quad z = a \text{ arc cosh } (u/a),$$

$$u = \frac{a}{2}\left(e^{z/a} + e^{-z/a}\right),$$

where u and v are defined as for the helicoid and where $u \geqq a$.† The two surfaces are applied when we choose as corresponding points those for which

$$u^2 = a^2 + u'^2, \quad v = v'.$$

The correspondence of the two surfaces when one is thus applied to the other has also been exhibited,‡ viz., the smallest circle of the catenoid corresponds to the axis of the helicoid, other circles corresponding to helices; the meridians of the catenoid correspond to the straight line generators of the helicoid.

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* Darboux, *Théorie générale des surfaces*, vol. 1, pp. 77, 82.

† The discussion is limited throughout to real points and real geodesics.

‡ Darboux, l. c., vol. 1, p. 83.

The Geodesics on the Catenoid.* The differential equation of the geodesics on a surface of revolution† becomes

$$(1) \qquad u^2 \frac{dv}{ds} = c, \quad ds^2 = dx^2 + dy^2 + dz^2.$$

If $c = 0$ in this equation we obtain, since $u = 0$ is not possible, $v =$ constant, so that the meridians on the catenoid are geodesics. Of the parallel circles only $u = a$ is a geodesic since it is the only one for which the principal normal is normal to the surface, this being a necessary and sufficient condition.

By substituting in ds in equation (1) the values of dx, dy, dz from the equations of the catenoid, we obtain

$$(2) \qquad v = c \int \frac{du}{\sqrt{(u^2 - a^2)(u^2 - c^2)}}.$$

There are three cases of this equation to discuss, viz.,

$$c > a, \quad c < a, \quad c = a,$$

in which only positive values of c will be considered since changing the sign of c only changes the sign of v in (1).

First case: $c > a$. Since $u^2 \geqq a^2$, it follows that $u^2 > c^2$ if the integral be real. Substituting $u = c/\sin \phi$, equation (2) reduces to

$$(3) \qquad v = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 . \sin^2\phi}} + m,$$

in which $k = a/c$ and m is any constant; m can be chosen equal to zero without loss of generality since the geodesics for the different values of m can be made to coincide by a rotation about the z-axis. Thus

$$u = \frac{c}{\sin \phi}, \quad v = F(k, \phi), \qquad 0 \leqq \phi \leqq \pi$$

are the equations of one branch of any geodesic of the first kind. But $\sin \phi = c/u = \cos \alpha$ from Clairaut's equation,‡ proved by writing (1) in the

* Cf. Darboux, l. c., vol. 3, p. 4, for a brief discussion of the geodesics on a surface of revolution having meridians extending to infinity and a parallel of minimum radius.

† Knoblauch, *Theorie der krummen Flächen*, pp. 147,148.

‡ Knoblauch, *l. c.*, p. 148.

form $u \cdot \dfrac{udv}{ds} = c$, in which a is the angle between an arc of the geodesic and the arc of the circle. It follows that

$$\phi = 2n\pi + \tfrac{1}{2}\pi \pm a,$$

where n is any integer.

But on the circle $u = c$, supposed above the xy-plane, $\sin \phi = \pm 1$ and therefore $a = n\pi$. The geodesic is therefore perpendicular to the meridian at the point where it is crossed by the circle $u = c$. Let u be positive and increase from c. Suppose ϕ to start from $\pi/2$ and to approach zero; then u increases indefinitely, and the geodesic approaches asymptotically the meridian $v = 0$. Considerations of symmetry at once show that the geodesic is symmetrical with respect to the meridian plane, $v = F(k, \pi/2)$, since u remains the same for both $\pi/2 + \phi$ and $\pi/2 - \phi$. A branch of the geodesic for a given c is thus enclosed between the planes $v = F(k, 0) = 0$ and $v = 2\,F(k, \pi/2) = F(k, \pi)$ and lies above the circle $u = c$ since $u > c$ along the geodesic. As ϕ increases from π to 2π another branch, equal to the one just described, is obtained which is tangent to the circle $u = c$ at $v = F(k, 3\pi/2)$. The equations of this branch may be written, since u and c are considered positive,

$$-u = c \sin \phi, \quad v = F(k, \phi), \qquad \pi \leqq \phi \leqq 2\pi.$$

This process may be continued indefinitely.

If $2F(k, \pi/2)$ is not commensurable with 2π the branches of the geodesic for a given c are repeated in endless procession without returning to the starting point. If the ratio of $2F(k, \pi/2)$ to 2π is equal to p/q where p and q are integral, then while v describes p complete revolutions the geodesic consisting of q branches returns to the starting point and may be said to be "closed" although each of its branches runs to infinity. But $2F(k, \pi/2)$ is a continuous function of k, varying monotonically from π to ∞ as k varies from 0 to 1. There must therefore be infinitely many values of k for which $2F(k, \pi/2)$ is commensurable with 2π. Consequently "closed" geodesics exist on every catenoid.

Moreover, the angle $2F(k, \pi/2)$ becomes infinite as c approaches a and k approaches unity, so that as the circle, $u = c$, approaches the smallest one, $u = a$, the geodesic winds about the catenoid an increasing number of times before it becomes tangent to $u = c$. But as c increases, $2F(k, \pi/2)$ decreases

and approaches its minimum π as c becomes infinite and k approaches zero. The limiting "closed" geodesic (not a circle) therefore winds about the catenoid just once and goes to infinity once.

SECOND CASE: $c < a$. By the substitution $u = a/\sin\phi$, equation (2) reduces to

$$v = k\int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} + m, \tag{4}$$

where $k = c/a$ and, as before, m may be taken equal to zero, and where we study the behaviour of u and v for values of ϕ in the interval $0 \leqq \phi \leqq \pi$. It is to be noticed that ϕ no longer has the geometrical interpretation of case one. Thus, for this branch,

$$u = \frac{a}{\sin\phi}, \quad v = k\, F(k, \phi), \qquad 0 \leqq \phi \leqq \pi$$

are the equations of the geodesic.

If $\phi = \pi/2$, then $u = a$ and $\alpha = \text{arc}\cos(c/a)$ since $\cos\alpha = c/u$. As ϕ increases to π, u increases without limit, and α increases to $\pi/2$, so that the geodesic approaches a meridian asymptotically. Meanwhile v is increased by $kF(k, \pi/2)$.

But the surface is symmetrical with respect to each meridian plane, and also with respect to the plane of the circle, $u = a$. As v, therefore, varies from 0 to $2\,kF(k, \pi/2) = kF(k, \pi)$, a branch of the curve starts at infinity on the upper or lower part of the surface at the plane $v = 0$, crosses the circle $u = a$ at an angle $\alpha = \text{arc}\cos(c/a)$ at the point P in the plane $v = kF(k, \pi/2)$, and passes to infinity on the lower or upper part of the surface where it approaches asymptotically the meridian $v = kF(k, \pi)$. There is a pencil of geodesics through this point P, each one intersecting $u = a$ in its own angle corresponding to the values of c. As v increases again by $kF(k, \pi)$, another branch of the geodesic is obtained crossing the circle $u = a$ at the plane $v = k\,F(k, 3\pi/2)$; so we may proceed indefinitely.

But as c approaches a and k approaches unity, $kF(k, \pi)$ becomes infinite and the geodesic winds about the catenoid increasingly often before crossing $u = a$, and the angle of intersection with $u = a$ approaches zero. As c and

k approach zero, $kF(k, \pi)$ approaches zero, and thus the limiting case of this class of geodesics is a meridian. If $2kF(k, \pi/2)$ is commensurable with 2π, the geodesic returns to its starting point and may be called "closed", by which is meant however only that for a given c the geodesic consists of a finite number of branches. There are an infinite number of values of k for which this is true. If it is not commensurable with 2π, the branches are repeated in endless procession and never return to the starting point.

THIRD CASE: $c = a$. If in equation (4) we put $k = c/a = 1$, the equation of the geodesics reduces to

$$(5) \qquad u = \frac{a}{\sin\phi}, \quad v = \int_0^\phi \frac{d\phi}{\cos\phi} = \log \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right).$$

But

$$\tanh v = \frac{e^v - e^{-v}}{e^v + e^{-v}}$$

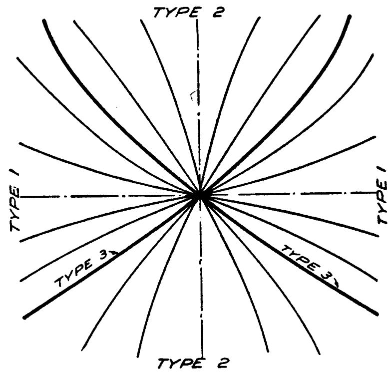
and, by substituting for v from (5),

$$\tanh v = \frac{2\tan\frac{\phi}{2}}{\sec^2\frac{\phi}{2}} = \text{in } \phi = \frac{a}{u}$$

Therefore along the geodesic, $u = a \coth v = a\,\dfrac{e^v + e^{-v}}{e^v - e^{-v}}$. This equation in the polar coordinates u, v in the xy-plane represents a cylinder whose intersection with the catenoid will give the geodesics for $c = a$. As v approaches zero, u becomes infinite, and as v increases indefinitely u decreases to a. The geodesic thus starts tangent asymptotically to the meridian, $v = 0$, comes into the finite part of the surface as v increases, and with constantly decreasing radius vector winds about the circle $u = a$ as v increases indefinitely. It is the limiting case of the geodesics of classes one and two.

Through a given point on the surface there will then pass a pencil of geodesics of type one ($c > a$), *a pencil of geodesics of type two* ($c < a$), *and two geodesics of type three* ($c = a$), *so situated that the two geodesics of type three separate those of type one from those of type two.* This follows immediately from Clairaut's equation, $u \cos a = c$. For a given point, u is

fixed and as c decreases continuously from u to zero through a, the angle α increases continuously from zero to $\pi/2$.



The Geodesics on the Helicoid. As has been said, the correspondence between the helicoid

$$x = u' \cos v', \qquad y = u' \sin v', \qquad z = av'$$

and the catenoid

$$x = u \cos v, \qquad y = u \sin v, \qquad z = a \text{ arc cosh } (u/a)$$

in the application of these surfaces is given by

$$u^2 = a^2 + u'^2, \qquad v = v'.$$

To a meridian $v = v_0 + 2n\pi$ of the catenoid correspond parallel straight line generators

$$x = u' \cos v_0, \qquad y = u' \sin v_0, \qquad z = a(v_0 + 2n\pi)$$

of the helicoid.

To the minimum parallel circle, $u = a$, of the catenoid, corresponds $u' = 0$, the axis of the helicoid, but to each point of the circle correspond on this axis an infinite number of points, $2\pi a$ units apart.

To any other parallel of the catenoid $u = b > a$ corresponds a helix $u' = \sqrt{b^2 - a^2}$ on the catenoid, but to each point of the circle correspond an infinite number of points on the helix lying in a line parallel to the z-axis (vertical) at a distance $2\pi a$ units apart. The correspondence of points in the

two surfaces may be made more precise by considering only those points of the catenoid on one side of the minimum parallel, for example only points for which $z \geqq 0$, and by supposing $u > 0$; then to each pair of values of u and v corresponds a single point of the surface, and to each point of the surface, only one value of u, and values of v differing by multiples of 2π.

If in the equations of the helicoid we suppose $u' > 0$, we shall have a single point corresponding to each pair of values of u' and v', and conversely, to each point of the surface one value of u', and one value of v'. So that both u and u' being positive we have to one point of the helicoid a single corresponding point of the catenoid, but to each point of the latter correspond points of the helicoid in the same vertical line, $2\pi a$ units apart. It is on the basis of such a correspondence that the equations of the geodesics of the helicoid are given in terms of elliptic functions. But in the discussion of the geometrical properties of the geodesics of the second class — those crossing the minimum parallel of the catenoid and the axis of the helicoid — the continuations of these lines have been included, namely points of the catenoid for which z is negative, and points of the helicoid for which u' is negative.

If the double signs are omitted from those equations they will indeed be still the equations of geodesic lines, but the correspondence would perhaps be not so clearly exhibited.

Each branch of a geodesic on the catenoid for the first case, $c > a$, was found between meridians making an angle of $F(k, \pi)$ with each other. They are tangent to a certain circle, $u = c$, at a point half way between two such meridians. A corresponding helicoid geodesic lies between planes which pass through the axis and differ in angle by $F(k, \pi)$, as appears from the geometrical meanings of v and v'. The geodesic approaches asymptotically the generators of the helicoid which lie in these planes and touches a certain helix corresponding to $u = c$. There is a pencil of geodesics of this type through each point of the surface.

If $F(k, \pi)$ is commensurable with 2π, other branches of the same geodesic for the same c are tangent to the same helix vertically over the first point of tangency; in other words there are equal geodesics, tangent to the same helix at points in a vertical line, which approach asymptotically the same generator. If not commensurable this is not the case. But at the limit, when $c = a$ and $F(k, \pi)$ is infinite, the geodesic of the third class winds about the helicoid indefinitely and only approaches the axis $u' = 0$. Moreover as c is indefinitely

increased, $F(k, \pi)$, the angle between the asymptotic generators, approaches its minimum value π.

The equations of these geodesics may be written in terms of elliptic functions. The equations of the corresponding catenoid geodesics are

$$\pm u \sin \phi = c, \qquad v = F(k, \phi), \qquad k = \frac{a}{c}, \qquad \phi = amv.$$

Therefore

$$\pm u \operatorname{sn} v = c, \qquad \text{or} \qquad \sqrt{u'^2 + a^2} \cdot \operatorname{sn} v = c = a/k.$$

Hence

$$u'^2 \cdot \operatorname{sn}^2 v = \frac{a^2}{k^2}(1 - k^2 \operatorname{sn}^2 v), \qquad \text{or} \qquad u' = \pm \frac{a \operatorname{dn} v'}{k \operatorname{sn} v'}$$

since $v = v'$. Thus the required equations are

$$x = u' \cos v' = \pm \frac{a \operatorname{dn} v' \cdot \cos v'}{k \operatorname{sn} v'},$$

$$y = u' \sin v' = \pm \frac{a \operatorname{dn} v' \cdot \sin v'}{k \operatorname{sn} v'},$$

$$z = av'.$$

We suppose u' positive throughout, so that the signs of x and y, in the equations above, are to be respectively those of $\cos v'$ and $\sin v'$.

For the second case, $c < a$, a branch of the catenoid geodesic crosses the circle $u = a$ midway between two meridians making an angle of $kF(k, \pi)$ with each other and is asymptotic to these meridians. Hence the branch of the corresponding helicoid geodesic crosses the axis at an angle $\alpha = \text{arc} \cos (c/a)$ midway between two generators asymptotic to the geodesic and making an angle of $v' = kF(k, \pi)$ with each other. If $kF(k, \pi)$ is commensurable with 2π there are equal branches of geodesics issuing from points on the axis distant $2n\pi a$ with parallel tangents at those points with a common asymptotic generator. If $kF(k, \pi)$ is not commensurable with 2π, this is not the case. There are an infinite number of values of k for which $kF(k, \pi)$ is commensurable with 2π.

The equation of the geodesics of this class may be written as follows:

$$x = u' \cos v' = \pm \frac{a \cos v'}{\operatorname{tn}(v'/k)},$$

$$y = u' \sin v' = \pm \frac{a \sin v'}{\operatorname{tn}(v'/k)},$$

$$z = av',$$

where $u' = \dfrac{a}{\operatorname{tn}(v'/k)}$, and $k = a/c$ from the equations of the corresponding catenoid geodesics. As in the first case, the sign of u' being throughout taken as positive, the signs of x and y are respectively those of $\cos v'$ and $\sin v'$.

The catenoid geodesics for the third case $c = a$ were found when projected on the xy-plane to be

$$u \tanh v = a,$$

which for the helicoid reduces to

$$u' (e^{v'} - e^{-v'}) = 2a,$$

since $u^2 = u'^2 + a^2$. If v' approaches zero, u' increases indefinitely; and as v' increases u' decreases and approaches zero. Therefore, as v' increases from 0 to 2π, the corresponding geodesic starting at an infinite distance traverses the entire helicoid and at each period of 2π lies closer to the axis.

The equations of the geodesics of this class may be written in the form

$$x = u' \cos v' = a \operatorname{cosech} v' \cos v',$$

$$y = u' \sin v' = a \operatorname{cosech} v' \sin v',$$

$$z = av',$$

where $u' = a \operatorname{cosech} v'$. Through each point of the helicoid, not on the axis, pass two geodesics of this class, separating, as in the catenoid, those of the first and second classes which pass through that point.

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